Exercise 1.1.8

Given $\sum_{n=1}^{1{,}000} n^{-1} = 7.485\,470\ldots$ set upper and lower bounds on the Euler-Mascheroni constant.

$$ANS. \quad 0.5767 < \gamma < 0.5778.$$

Solution

The definition of the Euler-Mascheroni constant is given on page 7 in Equation 1.13.

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^{-1} - \ln n\right) \tag{1.13}$$

Begin with Equation (1.9) on page 6, which gives a lower bound and an upper bound for the remainder of an infinite sum.

$$\int_{N+1}^{\infty} f(x)\,dx \le \sum_{n=N+1}^{\infty} a_n \le \int_{N+1}^{\infty} f(x)\,dx + a_{N+1} \tag{1.9}$$

Since $\sum_{n=1}^{1{,}000} n^{-1} = 7.485470860550345\ldots$, set $N = 1{,}000$ in the formula.

$$\int_{1{,}001}^{\infty} \frac{1}{x}\,dx \le \sum_{m=1{,}001}^{\infty} \frac{1}{m} \le \int_{1{,}001}^{\infty} \frac{1}{x}\,dx + \frac{1}{1{,}001}$$

Add $\sum_{m=1}^{1{,}000} m^{-1}$ to all sides.

$$\sum_{m=1}^{1{,}000} \frac{1}{m} + \int_{1{,}001}^{\infty} \frac{1}{x}\,dx \le \sum_{m=1}^{1{,}000} \frac{1}{m} + \sum_{m=1{,}001}^{\infty} \frac{1}{m} \le \sum_{m=1}^{1{,}000} \frac{1}{m} + \int_{1{,}001}^{\infty} \frac{1}{x}\,dx + \frac{1}{1{,}001}$$

$$\sum_{m=1}^{1{,}000} \frac{1}{m} + \int_{1{,}001}^{\infty} \frac{1}{x}\,dx \le \sum_{m=1}^{\infty} \frac{1}{m} \le \sum_{m=1}^{1{,}000} \frac{1}{m} + \int_{1{,}001}^{\infty} \frac{1}{x}\,dx + \frac{1}{1{,}001}$$

$$\sum_{m=1}^{1{,}000} \frac{1}{m} + \lim_{n\to\infty}\int_{1{,}001}^{n} \frac{1}{x}\,dx \le \lim_{n\to\infty}\sum_{m=1}^{n} \frac{1}{m} \le \sum_{m=1}^{1{,}000} \frac{1}{m} + \lim_{n\to\infty}\int_{1{,}001}^{n} \frac{1}{x}\,dx + \frac{1}{1{,}001}$$

$$\sum_{m=1}^{1{,}000} \frac{1}{m} + \lim_{n\to\infty} \ln x\Big|_{1{,}001}^{n} \le \lim_{n\to\infty}\sum_{m=1}^{n} \frac{1}{m} \le \sum_{m=1}^{1{,}000} \frac{1}{m} + \lim_{n\to\infty} \ln x\Big|_{1{,}001}^{n} + \frac{1}{1{,}001}$$

$$\sum_{m=1}^{1{,}000} \frac{1}{m} + \lim_{n\to\infty} \ln n - \ln 1{,}001 \le \lim_{n\to\infty}\sum_{m=1}^{n} \frac{1}{m} \le \sum_{m=1}^{1{,}000} \frac{1}{m} + \lim_{n\to\infty} \ln n - \ln 1{,}001 + \frac{1}{1{,}001}$$

Subtract $\lim_{n\to\infty} \ln n$ from all sides.

$$\sum_{m=1}^{1{,}000} \frac{1}{m} - \ln 1{,}001 \leq \lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{m} - \lim_{n\to\infty} \ln n \leq \sum_{m=1}^{1{,}000} \frac{1}{m} - \ln 1{,}001 + \frac{1}{1{,}001}$$

$$\sum_{m=1}^{1{,}000} \frac{1}{m} - \ln 1{,}001 \leq \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \ln n \right) \leq \sum_{m=1}^{1{,}000} \frac{1}{m} - \ln 1{,}001 + \frac{1}{1{,}001}$$

$$\sum_{m=1}^{1{,}000} \frac{1}{m} - \ln 1{,}001 \leq \gamma \leq \sum_{m=1}^{1{,}000} \frac{1}{m} - \ln 1{,}001 + \frac{1}{1{,}001}$$

Evaluate the left and right sides.

$$(7.485470860550345\ldots) - (6.90875477931522\ldots)$$

$$\leq \gamma \leq (7.485470860550345\ldots) - (6.90875477931522\ldots) + (0.000999000999000999\ldots)$$

Therefore,

$$0.5767160812351246\ldots \leq \gamma \leq 0.5777150822341257\ldots.$$